

Higher Angular Momentum Mixing in a Non-spherical Color Superconductor with Time Reversal Invariance Violation

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The angular momentum mixing in a non-spherical CSC with nonzero azimuthal quantum number and therefore violating time reversal invariance has been examined. The mixing is bound to occur because of the equal strength of the pairing potential mediated by one-gluon exchange for all partial waves to the leading order QCD running coupling constant and the nonlinearity of the gap equation. The free energy with mixing is lower than that with p -wave pairing only, but still higher than that of the spherical CSL state.

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I. INTRODUCTION

It has long been expected that quark matter at high baryon density and low temperature becomes a color superconductor(CSC)[1, 2]. CSC is characterized by a diquark condensate, which is analogous to the Cooper pair in an ordinary superconductor but the structure of the condensate is much richer because of the nonabelian color and flavor charges.

For very large baryon density, where the masses of u,d and s quarks can be ignored, the ground state is in the color-flavor-locked(CFL) phase[3]. The situation becomes more involved in moderate density because of the strange quark mass, β equilibrium and the charge neutrality conditions, which will induce a substantial Fermi momentum mismatch among different quark flavors and thereby reduce the available phase space for Cooper pairing. A number of exotic CSC phases have been proposed in the presence of mismatch, but a consensus point of view of the true ground state has not been reached[4, 5, 6, 7]. An interesting alternative in this circumstance is the single flavor pairing, which is obviously free from the Fermi momentum mismatch. Since the attractive interaction of quarks is provided in the antisymmetric color-antitriplet channel, the total angular momentum channel must be symmetric in order to insure the overall antisymmetric of the Cooper pair wave function as required by Pauli principle. Therefore, Cooper pair in single flavor pairing should be implemented at a higher total angular momentum, the obvious choice is the p -wave pairing($J=1$), which had been extensively explored in the literatures[8, 9, 10, 11].

The energy gaps in the single flavor pairing can be divided into two categories, spherical gaps (e.g. CSL) and non-spherical ones (e.g. polar phase and A phase). While a spherical gap is made of p -wave alone, a non-spherical gap may contain higher partial waves because of the nonlinearity of the gap equation. It was argued in the literature that the contribution of higher partial waves is of higher order in g with g the running coupling constant of QCD. This, however, is not the case, as we shall explain.

In a previous paper[12], we considered the single flavor CSC with longitudinal pairing, in which the pairing two quarks have equal helicity. Because of the equal strength of the pairing potential mediated by one-gluon exchange for all partial waves to the leading order QCD running coupling constant, a non-spherical pairing receives contributions from all odd J 's at the *same* order of g . A consistent solution to the gap equation with a definite azimuthal quantum number M , however, can be constructed. For the CSC that pairs red and green quarks only, the energy gap with a definite M is of the form

$$\Delta = e^{iM\varphi} \Delta_0 f(\cos\theta) \quad (1)$$

with θ and φ the polar angles of the relative momentum \vec{p} of the pairing quarks, where

$$f(\cos\theta) = \sum_{J \geq |M|, J=\text{odd}} b_J P_J^M(\cos\theta) \quad (2)$$

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with $P_J^M(x)$ the associate Legendre function. The free energy will be brought down by the mixing in comparison with that reported in the literature. For $M = 0$, we found that

$$f(\cos \theta) = e^{-17/3} [0.9866 P_1(\cos \theta) - 0.0673 P_3(\cos \theta) + 0.0214 P_5(\cos \theta) + \dots]. \quad (3)$$

The magnitude of the condensation energy is raised by 3.5 percent, which is too small to compete with the spherical color-spin-locked(CSL) state in single CSC. As we pointed out in [13], the energy balance between the CSL and non-spherical states maybe offset by the presence of anisotropy or the violation of time reversal invariance. In a compact star, the magnetic field and the stellar revolution provides such an opportunity.

In this paper, we extend the analysis in [12] to case with a nonzero azimuthal quantum number ($M \neq 0$). We will firstly derive the nonlinear integral equation for the angular dependence of the gap function and then give the numerical solution to this equation. Finally, a summary and concluding remarks will be given.

II. THE INTEGRAL EQUATION OF THE ANGULAR DEPENDENCE

The non-zero azimuthal quantum number violate the time reversal invariance and the gap function is proportional to $e^{i\varphi}$ in this case[14]. The condensation energy density for single flavor CSC with longitudinal pairing reads[12]

$$F = - \frac{3\bar{g}^2 \mu^4}{32\pi^4} \int d\nu \int d\nu' \int d^2\hat{p} \int d^2\hat{p}' V(\nu - \nu', \Theta) \times \frac{\phi(\nu, \hat{p})\phi(\nu', \hat{p}')}{\sqrt{[\nu^2 + |\phi(\nu, \hat{p})|^2][\nu'^2 + |\phi(\nu', \hat{p}')|^2]}} \\ + \frac{2\mu^2}{(2\pi)^3} \int d\nu \int d^2\hat{p} \left[|\nu| - \frac{\nu^2}{\sqrt{\nu^2 + |\phi(\nu, \hat{p})|^2}} \right] \quad (4)$$

where ν and ν' are Euclidean energies, Θ is the angle between \hat{p} and \hat{p}' and V is the pairing potential mediated by the one-gluon-exchange of QCD

$$V(\nu - \nu', \Theta) = D_l(\nu - \nu', \Theta) + D_t(\nu - \nu', \Theta) \quad (5)$$

with D_l and D_t is the longitudinal(color electric) and transverse(color magnetic) part of hard-dense-loop(HDL) gluon propagator respectively. The gap function $\phi(\nu, \hat{p})$ is extracted from the Nambu-Gorkov off diagonal blocks of the quark self-energy and its value at $\nu = 0$ gives Δ of (1). The gap equation can be derived by minimizing the condensation energy with respect to the gap function

$$\frac{\delta F}{\delta \phi} = 0 \quad (6)$$

and we end up with

$$\phi(\nu, \hat{p}) = \frac{g^2 \mu^2}{24\pi^3} \int d\nu' \int d^2\hat{p}' V(\nu - \nu', \theta) \frac{\phi(\nu', \hat{p}')}{\sqrt{\nu'^2 + |\phi(\nu', \hat{p}')|^2}} \quad (7)$$

A consistent derivation of the gap equation up to the subleading order need to include the one-loop self energy of quarks, the net result is to replace the first term in the square root on RHS of Eq.(7) by $\nu'^2/Z^2(\nu')$ with $Z(\nu')$ the wave function renormalization factor[15, 16]. But it will not interfere with the angular dependence of the gap function as we have seen in[12].

The potential V can be expanded in terms of Legendre polynomials[15]

$$V(\nu - \nu', \Theta) = \frac{1}{6\mu^2} \ln \frac{\omega_c}{|\nu - \nu'|} \sum_{J=0}^{\infty} (2J+1) P_J(\cos \Theta) + \frac{1}{2\mu^2} \sum_{J=1}^{\infty} (2J+1) c_J P_J(\cos \Theta) \quad (8)$$

with $\omega_c = \frac{1024\sqrt{2}\pi^4\mu}{g^5}$ and c_J given by

$$c_J = -2 \sum_{n=1}^J \frac{1}{n}. \quad (9)$$

From this expansion, we can see that the pairing strength are equal to leading order of the QCD running coupling constant, but subleading terms fall off with increasing J . It is this falling off that makes the amount of the angular momentum mixing numerically small for the solutions in[12] and in this letter as we shall see below.

Substituting Eq. (8) into the gap equation (7), we have

$$\begin{aligned} \phi(\nu, \hat{p}) = & \bar{g}^2 \int_0^{\omega_0} d\nu' \left\{ \frac{1}{2} \left(\ln \frac{\omega_c}{|\nu - \nu'|} + \ln \frac{\omega_c}{\nu + \nu'} \right) \frac{\phi(\nu', \hat{p})}{\sqrt{\nu'^2 + |\phi(\nu', \hat{p})|^2}} \right. \\ & \left. + \frac{3}{2\pi} \int d\hat{p}' \frac{1}{1 - \hat{p} \cdot \hat{p}'} \left[\frac{\phi(\nu', \hat{p}')}{\sqrt{\nu'^2 + |\phi(\nu', \hat{p}')|^2}} - \frac{\phi(\nu', \hat{p})}{\sqrt{\nu'^2 + |\phi(\nu', \hat{p})|^2}} \right] \right\} \end{aligned} \quad (10)$$

where $\bar{g}^2 = g^2/(18\pi^2)$ and a UV cutoff $\omega_0 \sim g\mu$ is introduced. On writing

$$\phi(\nu, \hat{p}) = e^{iM\varphi} \psi(\nu, \cos \theta), \quad (11)$$

the equation for the real function $\psi(\nu, \theta)$ reads

$$\begin{aligned} \psi(\nu, x) = & \bar{g}^2 \int_0^{\omega_0} d\nu' \left\{ \frac{1}{2} \left(\ln \frac{\omega_c}{|\nu - \nu'|} + \ln \frac{\omega_c}{\nu + \nu'} \right) \times \frac{\psi(\nu', x)}{\sqrt{\nu'^2 + \psi^2(\nu', x)}} \right. \\ & \left. + 3 \int_{-1}^1 dx' \left[\frac{\psi(\nu', x') I_M(x, x')}{\sqrt{\nu'^2 + \psi^2(\nu', x')}} - \frac{1}{|x - x'|} \frac{\psi(\nu', x)}{\sqrt{\nu'^2 + \psi^2(\nu', x)}} \right] \right\} \end{aligned} \quad (12)$$

where $x = \cos \theta$ and

$$I_M(x, x') = \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \frac{e^{iM(\varphi' - \varphi)}}{1 - \hat{p} \cdot \hat{p}'} = \text{real} \quad (13)$$

The most favored pairing channel corresponds to $|M| = 1$ and there is no difference between M and $-M$. We only consider the case of $M = 1$ below. We have $I_1(x, x') = \mathcal{I}(x, x')/|x - x'|$ with \mathcal{I} given by

$$\mathcal{I}(x, x') = \frac{1 - xx' - |x - x'|}{\sqrt{(1 - x^2)(1 - x'^2)}} \quad (14)$$

The gap equation (12) can be further simplified by using the approximation in[8]

$$\ln \frac{\omega_c}{|\nu - \nu'|} \simeq \ln \frac{\omega_c}{|\nu_{>}|} \quad (15)$$

with $|\nu_{>}| = \max(|\nu|, |\nu'|)$. Introducing new variables

$$\xi = \ln \frac{\omega_c}{\nu}, \quad a = \ln \frac{\omega_c}{\omega_0} \quad (16)$$

and writing $\psi(\nu, x)$ as $\psi(\xi, x)$, we have

$$\psi(\xi, x) = \xi \Phi(\xi, x) - \int_a^\xi d\xi' \xi' \frac{d\Phi}{d\xi'} + 3 \int_{-1}^1 dx' \frac{\Phi(a, x') \mathcal{I}(x, x') - \Phi(a, x)}{|x - x'|} \quad (17)$$

where $\Phi(\xi, x)$ is defined by

$$\Phi(\xi, x) \equiv \bar{g}^2 \int_\xi^\infty d\xi' \frac{\psi(\xi', x)}{\sqrt{1 + \frac{\psi^2(\xi', x)}{\omega_c^2} e^{2\xi'}}} \quad (18)$$

Taking the first derivative of both sides with respect to ξ in Eq.(17), we have

$$\frac{d\psi(\xi, x)}{d\xi} = \Phi(\xi, x) \quad (19)$$

which implies

$$\frac{d\psi}{d\xi} \rightarrow 0 \quad (20)$$

as $\xi \rightarrow \infty$ for all x . Another derivative of (19) yields

$$\frac{d^2\psi(\xi, x)}{d\xi^2} + \frac{\bar{g}^2\psi(\xi, x)}{\sqrt{1 + \frac{\psi^2(\xi, x)}{\omega_c^2}}e^{2\xi}} = 0. \quad (21)$$

The x -dependence of $\psi(\xi, x)$ will be fixed by the Eq.(17) at $\xi = a$, i. e.

$$a\Phi(a, x) - \psi(a, x) + 3 \int_{-1}^1 dx' \frac{\Phi(a, x')\mathcal{I} - \Phi(a, x)}{|x - x'|} = 0 \quad (22)$$

The solution to (21) subject to the boundary condition (20) has been obtain in[12] up to subleading order, which applies to the case $M = 1$ as well. We have

$$\psi(a, x) \simeq \Delta_0 f(x) \left[\frac{\pi}{2} - \bar{g}(b - a) - \bar{g}\ln 2 \right] + O(g) \quad (23)$$

and

$$\Phi(a, x) = \bar{g}\Delta_0 f(x) + O(g) \quad (24)$$

where Δ_0 is the s-wave gap given by

$$\frac{\pi}{2} - \bar{g}\ln \frac{2\omega_c}{\Delta_0} = 0 \quad (25)$$

and $b = \ln \frac{\omega_c}{\Delta_0 |f(x)|}$. Substituting Eq.(23) and Eq.(24) into Eq.(22), we derive the integral equation for the angle dependent factor $f(x)$

$$f(x)\ln|f(x)| + 3 \int_{-1}^1 dx' \frac{f(x) - f(x')\mathcal{I}(x, x')}{|x - x'|} = 0 \quad (26)$$

III. THE SOLUTION TO THE INTEGRAL EQUATION

As we have done in[12], this type of integral equation can be solved by a variational method. Substituting the solution of (21) and (20) into (4), the condensation energy density becomes a functional of $f(x)$ (see appendix B in[12]),

$$F = \frac{\mu^2 \Delta_0^2}{2\pi^2} \mathcal{F}[f] \quad (27)$$

with

$$\mathcal{F}[f(x)] = \int_{-1}^1 dx f^2(x) \left[\ln|f(x)| - \frac{1}{2} \right] + \frac{3}{2} \int_{-1}^1 dx \int_{-1}^1 dx' \frac{f^2(x) - 2f(x)f(x')\mathcal{I} + f^2(x')}{|x - x'|} \quad (28)$$

It can be easily verified that the variational minimization of Eq.(28) does solve Eq.(26). Before the numerical calculations, we consider a trial function as

$$f(x) = cP_1^1(x) = c\sqrt{1 - x^2} \quad (29)$$

Substituting it into the condensation energy density Eq.(28) and the minimization with respect to c yields

$$c = \frac{1}{2} e^{5/6} e^{-6} \quad (30)$$

at which

$$\mathcal{F} = -5.863 \times 10^{-6} \quad (31)$$

This trial function is what people carried over from A phase in ^3He [10, 11], which contains p -wave only, but it is not optimal. The free energy will be lowered further by including higher partial waves as we shall see.

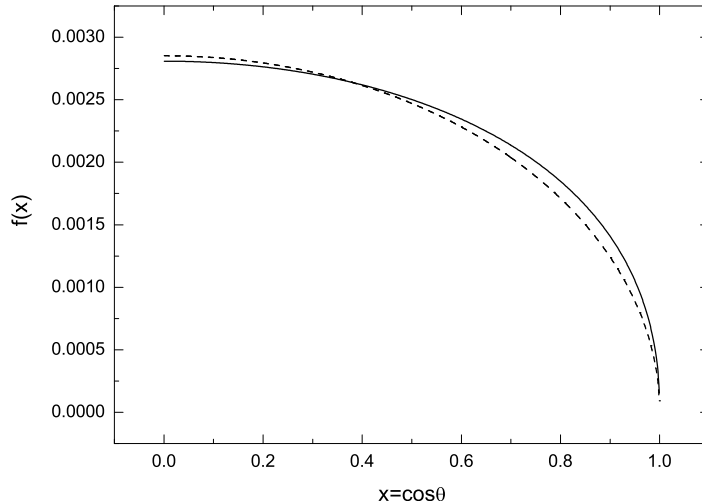


FIG. 1: The angular dependence of the gap function with nonzero azimuthal quantum number. The dashed line and solid one correspond to the trial function and the numerical solution to Eq.(26) respectively.

Following the same procedure in[12], we obtain the numerical solutions to Eq.(26). In Fig.1, we show the angular dependence of the gap function with nonzero azimuthal quantum number, the dashed line and the solid line are the trial function and the numerical solution to Eq.(26) respectively. They depart from each other slightly, indicating a small mixture of higher partial waves. We find the minimum value of the target functional with the solid line

$$\mathcal{F} = -5.977 \times 10^{-6} \quad (32)$$

which drop from Eq.(31) by 1.9 percent.

As we have found in[12], the drop of the condensation energy with longitudinal pairing with zero azimuthal quantum number by angular momentum mixing is numerically small. Here, with nonzero azimuthal quantum number, the situation is also true. With these small drop amount in condensation energy, the non-spherical pairing in single flavor CSC can not compete with the spherical pairing state CSL.

Regarding the angular momentum contents for our solutions according to Eq.(2) for $M = 1$, we found

$$f(\cos \theta) = \frac{1}{2} e^{5/6} e^{-6} [1.0112 P_1^1(\cos \theta) + 0.0212 P_3^1(\cos \theta) + 0.004 P_5^1(\cos \theta) + \dots]. \quad (33)$$

IV. CONCLUDING REMARKS

In summary, we have explored the angular momentum mixing in a non-spherical CSC with nonzero azimuthal quantum number, which violates the time reversal invariance because the gap function is not real in this case. The mixing is driven by the equal strength of the pairing potential mediated by one-gluon exchange for all partial waves to the leading order QCD running coupling constant and the nonlinearity of the gap equation. However, the gaining factor of the condensation energy caused by mixing is smaller than that by forming the spherical CSL state. The angular momentum mixing in the parallel case with transverse pairing has also been considered in [13], in which the amount of the free energy drop is also small. Therefore, we conjecture that angular momentum mixing of various non-spherical CSC is not sufficient to compete with the CSL energetically in the ultra relativistic limit. A rigorous proof of this statement is anticipating.

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